while in the case of waves generated by the waveguide, i.e. by the interval $(t_1, t_2)$, we have

$$\varepsilon(k) = -(b + O_1)\exp(-2kf_3), \qquad b > 0 \tag{9}$$

This method makes possible a generalization to the case of a multi-extremal function $c(z)$. The case of decaying waves when $c(z)$ is monotonous, was studied previously by V. Iu. Zavadskii. Under the quantum-mechanical treatment, such solutions describe quasi-stationary states [3].


BIBLIOGRAPHY

1. Alenitsyn, A. G., Rayleigh waves in a nonhomogeneous elastic slab. PMM Vol.28, No.5, 1964.
2. Alenitsyn, A. G., Rayleigh waves in a nonhomogeneous elastic half-space of waveguide type. PMM Vol.31, No.2, 1967.
3. Baz', A. I., Zel'dovich, Ia. B. and Perelomov, A. M., Scattering, reactions and decays in nonrelativistic quantum mechanics. M., Izd. "Nauka", 1966.


Translated by L. K.

# MOTION OF GAS BEHIND AN EXPANDING DETONATION WAVE IN SPACE WITH A CUT-OUT CONE

PMM Vol. 31, No. 5, 1967, pp. 916-928

Ia. M. KAZHDAN
(Moscow)

(Received February 16, 1967)

Motion of gas behind a detonation wave expanding from the point of ignition $O$ (coordinate origin) in a space filled with an explosive and with a cut-out hollow cone (axis of the cone: $x = 0$, $y = 0$, $z \leq 0$), possesses a cylindrical symmetry and is self-similar. Consequently, all gas-dynamic magnitudes are functions of two independent variables $\xi = r/t$, $\eta = z/t$, $r = \sqrt{x^2 + y^2}$ (here $t$ denotes time). These functions satisfy the gas-dynamic equations with the corresponding boundary conditions, written in terms of these variables. Numerical methods of solution of partial differential equations (in two independent variables $\xi$ and $\eta$) must however be used to obtain the above magnitudes.

S. K. Godunov assumed that a region exists on the $\xi\eta$-plane, where the flow coincides with the corresponding spherically symmetric flow obtained by Zel'dovich [1]. The latter flow occurs when a detonation wave expands from the origin $O$, the whole space being filled with an explosive. The motion of the gas is, in this case, spherically symmetric and self-similar, and determination of gas-dynamic functions reduces to the integration of a system of ordinary differential equations with the corresponding boundary conditions.

Our present investigations confirm the above assumption and give a method of determination of this region. In addition, asymptotics in the vicinity of singular points appearing in the solution, are given.

**1.** A flow behind a detonation wave can be defined by a system of gas-dynamic equations, an equation of state of the explosion products, and by initial and boundary conditions. Variables are chosen to suit the problem. The flow in question possesses cylindrical symmetry, hence it is sufficient to consider the motion in a semi-plane passing through the $z$-axis of symmetry and bounded by it. The $r$-axis passes through the point $O$ and is perpendicular to the $z$-axis. Moreover, since the flow is isentropic, gas-dynamic equations in terms of independent $(t, r, z)$-variables have the form

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_z \frac{\partial v_r}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial r} = 0, \qquad \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} + \frac{1}{\rho}\frac{\partial p}{\partial z} = 0$$

$$\frac{\partial \rho}{\partial t} + v_r \frac{\partial \rho}{\partial r} + v_z \frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial v_r}{\partial r} + \frac{\partial v_z}{\partial z} + \frac{v_r}{r}\right) = 0 \qquad (1.1)$$

while the equation of state of explosion products is $p = \varkappa p^{\varkappa}$.

Here $\rho$ denotes the density, $p$ the pressure, $\varkappa$ is the ratio of specific heats, while $v_r$ and $v_z$ are the relevant velocity components.

At the initial moment $t = 0$ of ignition of gas at the point $O$, all space exterior to the empty cone (whose vertex is at the origin, $z < 0$ and the angle between the axis and the generator is equal to $\gamma$) is filled with an immovable ($v_r = v_z = 0$) explosive of constant density $\rho_0 = \varkappa / \varkappa + 1$. Initial pressure $p_0 = 0$.

The resulting flow is bounded by the detonation wave front on one side and by a free surface ($p = 0$) on the other side.

Detonation wave front satisfies the Jouguet condition, consequently the values assumed by the functions at the wave front should satisfy

$$\rho_1 (D - u_1) = \frac{\varkappa}{\varkappa + 1} D, \qquad \rho_1 (D - u_1)^2 + p_1 = \frac{\varkappa}{\varkappa + 1} D^2$$

$$w_1 = 0, \qquad D = u_1 + c_1, \qquad c = \varkappa \rho^{1/2 (\varkappa - 1)} \qquad (1.2)$$

where $u$ and $w$ are the velocity components, which are, respectively, normal and tangential to the wave front, $D$ is the wave velocity and $c$ denotes the velocity of sound. Subscript 1 denotes the values at the wave front. We can easily solve (1.2) to obtain

$$\rho_1 = 1, \quad u_1 = 1, \qquad w_1 = 0, \qquad D = \varkappa + 1, \qquad c_1 = \varkappa$$

Thus we find that the wave velocity is constant at all points of the wave front. Consequently, the wave front appears, at the instant $t$ and in the semi-plane $rz$ (Fig. 1) as a

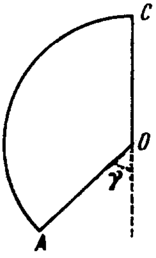



Fig. 1

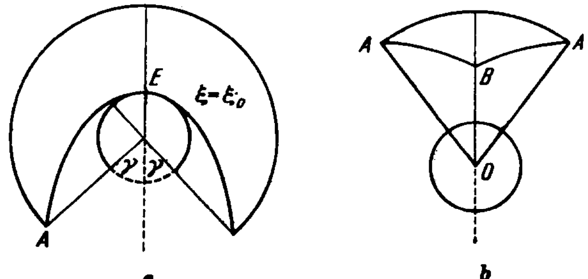



Fig. 2

circular arc $AC$ with the center at $O$ and of radius $Dt$ ; $A$ and $C$ are the points of intersection with the generator of the cone and with the $z$-axis respectively. The problem is self-similar, therefore we shall use self-similar variables $\xi = r/t$ and $\eta = z/t$. Taking $v_r$, $v_z$ and $c^2$ as unknown functions, we can write (1.1) in the form

$$\frac{\partial v_r}{\partial \xi}(v_r - \xi) + \frac{\partial v_r}{\partial \eta}(v_z - \eta) + \frac{1}{\varkappa - 1}\frac{\partial c^2}{\partial \xi} = 0$$

$$\frac{\partial v_z}{\partial \xi}(v_r - \xi) + \frac{\partial v_z}{\partial \eta}(v_z - \eta) + \frac{1}{\varkappa - 1}\frac{\partial c^2}{\partial \eta} = 0 \qquad (1.3)$$

$$\frac{\partial c^2}{\partial \xi}(v_r - \xi) + \frac{\partial c^2}{\partial \eta}(v_z - \eta) + (\varkappa - 1)\left(\frac{\partial v_r}{\partial \xi} + \frac{\partial v_z}{\partial \eta} + \frac{v_r}{\xi}\right)c^2 = 0$$

The arc $AC$ of the circle $\xi^2 + \eta^2 = D^2$ corresponds to the wave front, on which we have

$$v_r = \xi / D, \qquad v_z = \eta / D, \qquad c^2 = \varkappa^2$$

and we have exactly the same situation in the case of spherical symmetry.

Since the products of explosion escape into empty space, a rarefaction wave is originated centrally, with a vertex at the point $A$ and terminating at the free boundary $\eta = \eta(\xi)$ originating at $A$, along which $p = 0$. The principal part of the flow in this rarefaction wave corresponds to the Prandtl-Meyer solution, hence we can change to polar coordinates $\alpha$ and $\delta$ with the origin at $A$

$$\xi = D\sin\gamma + \alpha\cos(\gamma + \sqrt{h}\,\delta), \qquad \eta = -D\cos\gamma + \alpha\sin(\gamma + \sqrt{h}\,\delta) \qquad (1.4)$$

together with the corresponding velocities

$$v_r = D\sin\gamma + v_\alpha\cos(\gamma + \sqrt{h}\,\delta) - v_\delta\sin(\gamma + \sqrt{h}\,\delta) \qquad \left(h = \frac{\varkappa+1}{\varkappa-1}\right) \qquad (1.5)$$

$$v_z = -D\cos\gamma + v_\alpha\sin(\gamma + \sqrt{h}\,\delta) + v_\delta\cos(\gamma + \sqrt{h}\,\delta)$$

Here $\sqrt{h}\,\delta$ is an angle between the ray emerging from $A$ and the direction of the front at $A$, $\alpha$ is the radial distance from $A$, while $v_\alpha$ and $v_\delta$ are the radial (a ray emerging from $A$ at the angle $\sqrt{h}\,\delta$) and transverse velocity components. Using these variables and functions we can write the initial gas-dynamic system in the form

$$(v_\alpha - \alpha)\frac{\partial v_\alpha}{\partial \alpha} + \frac{v_\delta}{\alpha}\left(\frac{1}{\sqrt{h}}\frac{\partial v_\alpha}{\partial \delta} - v_\delta\right) + \frac{1}{\varkappa - 1}\frac{\partial c^2}{\partial \alpha} = 0$$

$$(v_\alpha - \alpha)\frac{\partial v_\delta}{\partial \alpha} + \frac{v_\delta}{\alpha}\left(\frac{1}{\sqrt{h}}\frac{\partial v_\delta}{\partial \delta} + v_\alpha\right) + \frac{1}{(\varkappa - 1)\sqrt{h}\,\alpha}\frac{\partial c^2}{\partial \delta} = 0 \qquad (1.6)$$

$$(v_\alpha - \alpha)\frac{\partial c^2}{\partial \alpha} + \frac{v_\delta}{\sqrt{h}\,\alpha}\frac{\partial c^2}{\partial \delta} + (\varkappa - 1)c^2\left[\frac{\partial v_\alpha}{\partial \alpha} + \frac{v_\alpha}{\alpha} + \frac{1}{\sqrt{h}\,\alpha}\frac{\partial v_\delta}{\partial \delta} + \right.$$
$$\left. + \frac{v_\alpha\cos(\gamma + \sqrt{h}\delta) - v_\delta\sin(\gamma + \sqrt{h}\delta) + D\sin\gamma}{\alpha\cos(\gamma + \sqrt{h}\delta) + D\sin\gamma}\right] = 0$$

Equation of the wave front is

$$\alpha = 2(\varkappa + 1)\sin\sqrt{h}\,\delta \left(0 \leqslant \delta \leqslant \frac{\pi - \gamma}{2\sqrt{h}}\right)$$

and at the front we have

$$v_\delta = \varkappa\cos\sqrt{h}\,\delta, \qquad v_\alpha = (\varkappa + 2)\sin\sqrt{h}\,\delta, \qquad c^2 = \varkappa^2$$

**2.** We can represent the solution of our problem as a perturbation of a spherically symmetric motion of a strong rarefaction wave caused by expansion of the explosion products into an empty space, following the detonation wave. We can infer from general considerations that the boundary between the region of spherically symmetric solution and the perturbed medium, should consist of a characteristic together with a shock wave front, should it occur. Obviously, the characteristic should originate at the point $A$ (Fig. 1), i.e. at the vertex of the rarefaction wave. The following differential equation defines this characteristic:

$$\frac{d\zeta}{d\delta} = \frac{\sqrt{h}\,\zeta a\,(D\sin\sqrt{h}\,\delta - y)}{cy - ax} \qquad (\zeta = \sqrt{\xi^2+\eta^2}) \tag{2.1}$$

$$a = \sqrt{(\zeta-u)^2 - c^2}, \quad x = D\cos\sqrt{h}\,\delta, \quad y = \begin{cases} \sqrt{\zeta^2 - x^2} & (0 \leqslant \delta \leqslant \delta_0) \\ -\sqrt{\zeta^2 - x^2} & (\delta \geqslant \delta_0) \end{cases}$$

and the initial conditions are

$$\zeta = (\varkappa + 1) \quad \text{when } \delta = 0$$

Here $\zeta$ is a self-similar variable of a spherically symmetric solution, while $(\delta_0, D\cos\sqrt{h}\delta_0)$ is the point of intersection of the required integral curve with the singular integral curve $y = 0$. Self-similar functions $u(\zeta)$ and $c(\zeta)$ of the spherically symmetric solution are defined by

$$\frac{du}{d\zeta} = \frac{2uc^2}{\zeta\,[(\zeta-u)^2 - c^2]} \tag{2.2}$$

$$\frac{dc}{d\zeta} = \frac{(\varkappa-1)\,uc\,(\zeta-u)}{\zeta\,[(u-\zeta)^2 - c^2]} \tag{2.3}$$

together with initial conditions

$$\zeta = (\varkappa+1), \qquad u = 1, \qquad c = \varkappa$$

The initial point ($\delta = 0$, $\zeta = \varkappa + 1$) is a singular point of (2.1). Curves $\zeta = \varkappa + 1$ define the detonation wave front and $y = 0$ are the integrals of this equation which, however, are not at the same time characteristics (the relation along the characteristics does not hold). Choice of a characteristic is accomplished with help of an initial asymptotic of the function $\zeta(\delta)$ obtained from (2.1) under the assumption that its solutions which we seek, differ from $\zeta = \varkappa + 1$ and $\zeta = x$

$$\zeta = \varkappa + 1 - {}^{64}/_{81}\,\varkappa h^2\delta^4 \tag{2.4}$$

It was shown in [1] that in the case of spherical symmetry, a characteristic $\zeta = \zeta_0$ exists which contains the quiescent region: when $\zeta \leq \zeta_0$, $u = 0$ and $c = \zeta_0$. This characteristic obviously satisfies (2.1). We find that when $\delta$ varies from 0 to some value $\delta_1$, $\zeta(\delta)$ varies monotonously from $\varkappa + 1$ to $\zeta_0$.

Indeed, spherical symmetry implies that when the point $A$ moves along the circle $\zeta = D$, the characteristic will rotate around $O$ in the manner of an inflexible line. If $\zeta(\delta)$ was not monotone, then the characteristics belonging to the same family would intersect, and this would contradict the self-similar solution of Zel'dovich. It was found by numerical integration that

$$\sqrt{h}\,\delta_1 < {}^1/_2\pi - \operatorname{arctg}\zeta_0/D \qquad (\zeta(\delta_1) = \zeta_0)$$

With $\delta > \delta_1$, the characteristic coincides, up to the moment of intersection with the $z$-axis, with the arc of the circle $\zeta = \zeta_0$. Obviously, for each value of $\varkappa$ there exists such a value of the apex angle $\gamma_0$ that, when $\gamma > \gamma_0$, then $\zeta > \zeta_0$ between the point $A$ and the $\eta$-axis, i.e. the characteristic $\zeta = \zeta_0$ is not reached. Fig. 2a and b show schematically the distribution of the characteristic in the $\zeta\eta$-plane relative to the angle $\gamma$. We note that in $\alpha$ and $\delta$ coordinates, equation of the characteristic becomes

$$\alpha = \varphi(\delta) = D\sin\sqrt{h}\delta - y \tag{2.5}$$

**3.** The problem is two-dimensional in the perturbed region, therefore in the following the gas-dynamic functions sought will be $f = c^2$, $v_\alpha$ and $v_\delta$.

In the region where the solution is spherically symmetric, $v_\alpha$. and $v_\delta$ are related to the velocity $u$ as follows:

$$v_\alpha = \frac{u}{\zeta} + D \sin \sqrt{h}\,\delta \left(1 - \frac{u}{\zeta}\right), \qquad v_\delta = D\left(1 - \frac{u}{\zeta}\right)\cos \sqrt{h}\,\delta \tag{3.1}$$

Values of the functions in the perturbed region along the characteristic determined previously should coincide with those given by the spherically symmetric solution. Consequently, from Formulas (2.3) to (2.5) and (3.1) it follows for small $\delta$, that along this line the following asymptotic should be valid

$$v_\alpha \approx \varkappa \sqrt{h}\left[1 + \left(\frac{208}{81(\varkappa+1)} - \frac{1}{6}\right) h\delta^2\right]\delta, \qquad v_\delta \approx \varkappa\left[1 + \left(\frac{16}{9(\varkappa+1)} - \frac{1}{2}\right)h\delta^2\right]$$
$$f \approx \varkappa^2\left[1 - \frac{16}{9}\,\delta^2\right] \qquad (f = c^2), \qquad \alpha \approx \frac{64}{81}\varkappa h^{3/2}\delta^3 \tag{3.2}$$

Solution of the system defining the principal term in the perturbed region near the point $A$ (the Prandtl-Meyer flow)

$$v_{\delta_0}\left(\frac{1}{\sqrt{h}}\, v_{\alpha_0}' - v_{\delta_0}\right) = 0, \qquad v_{\delta_0}\left(\frac{1}{\sqrt{h}}\, v_{\delta_0}' + v_{\alpha_0}\right) + \frac{1}{(\varkappa-1)\sqrt{h}}\, f_0' = 0$$
$$\frac{v_{\delta_0}}{\sqrt{h}}\, f_0' + (\varkappa - 1) f_0\left[v_{\alpha_0} + \frac{1}{\sqrt{h}}\, v_{\delta_0}'\right] = 0 \tag{3.3}$$

with initial conditions

$$v_{\alpha_0} = 0, \qquad v_{\delta_0} = \varkappa, \qquad f = \varkappa^2 \qquad \text{when} \quad \delta = 0$$

has the form

$$v_{\alpha_0}(\delta) = \varkappa \sqrt{h} \sin\delta, \qquad v_{\delta_0}(\delta) = \varkappa\cos\delta, \qquad f_0(\delta) = \varkappa^2\cos^2\delta \tag{3.4}$$

Obviously, the asymptotic (3.2) coincides with the asymptotic of (3.4) only within the first term. This means that, in the immediate vicinity of the characteristic when $\alpha$ is of the order $\delta^3$, the solution is somewhat different in structure. We shall use $\delta$ and $\psi = \alpha\delta^{-3}$ as independent variables to find the asymptotic in this region and we shall write its expansion as

$$v_\alpha = v_{\alpha_0}(\psi)\,\delta + v_{\alpha_1}(\psi)\,\delta^3 + \ldots, \qquad v_\delta = v_{\delta_0}(\psi) + v_{\delta_1}(\psi)\,\delta^2 + \ldots$$
$$f = f_0(\psi) + f_1(\psi)\,\delta^2 + \ldots$$

Inserting these functions into (1.6) and equating to zero the coefficients of the powers of $\delta$, we obtain two systems of ordinary differential equations. Initial values of these functions are obtained on a characteristic corresponding to

$$\psi = \psi_0 = \frac{64}{81}\varkappa h^{3/2}$$

from the asymptotic (3.2). Thus the system and initial data denoted a subscript 0, are

$$v_{\delta_0}\left[\frac{1}{\sqrt{h}}\left(v_{\alpha_0} - 3v_{\alpha_0}'\psi\right) - v_{\delta_0}\right] + \frac{1}{\varkappa - 1} f_0' = 0, \qquad v_{\delta_0} v_{\delta_0}' + \frac{f_0'}{\varkappa - 1} = 0 \tag{3.5}$$
$$v_{\delta_0} f_0' + f_0 v_{\delta_0}' = 0, \qquad v_{\alpha_0} = \varkappa\sqrt{h}, \qquad v_{\delta_0} = \varkappa, \qquad f = \varkappa^2 \qquad (\psi = \psi_0)$$

Then $v_{\alpha_0} = \varkappa\sqrt{h}$, $v_{\delta_0} = \varkappa$, $f = \varkappa^2$ will be the solution of (3.5) with the relevant initial conditions. For a function with a subscript 1, the system and initial conditions are given by

$$\varkappa\left[\frac{3}{\sqrt{h}}\left(v_{\alpha_1} - v_{\alpha_1}'\psi\right) - v_{\delta_1}\right] + \frac{1}{\varkappa - 1} f_1'\psi = 0$$
$$2f_1 - 3f_1'\psi + \varkappa(\varkappa - 1)\left[2v_{\delta_1} - 3v_{\delta_1}'\psi\right] = -\varkappa^2(\varkappa + 1) \tag{3.6}$$

$$\varkappa^2(\varkappa-1)\sqrt{h}\,v_{\delta_1}'\psi-(\varkappa-1)\varkappa^2 v_{\alpha_1}'\psi+f_1'\psi\frac{6\varkappa v_{\delta_1}-3f_1-\varkappa^2 h}{\varkappa\sqrt{h}}=$$
$$=\frac{\varkappa^2}{h}\psi-\frac{2f_1(f_1-2\varkappa v_{\delta_1})}{\varkappa\sqrt{h}}$$
$$\psi=\psi_0,\quad v_{\alpha_1}=\varkappa h^{3/2}\left(\frac{203}{81(\varkappa+1)}-\frac{1}{6}\right),\quad v_{\delta_1}=\varkappa h\left(\frac{16}{9(\varkappa+1)}-\frac{1}{2}\right),\quad f_1=-\frac{16}{9}\varkappa^2$$

The system (3.6) has a first integral

$$f_1+\varkappa(\varkappa-1)v_{\delta_1}=-{}^1/_2\varkappa^2(\varkappa+1)+c\psi^{2/3} \tag{3.7}$$

and initial conditions imply that $c=0$.

First integral (3.7) and the substitution

$$v_{\alpha_1}=V_{\alpha_1}-\psi/\varkappa+1 \tag{3.8}$$

yield (3.6) in its reduced form

$$\frac{dV_{\alpha_1}}{df_1}=\frac{4hf_1(f_1+\varkappa^2)+(3f_1+5\varkappa^2)[6\varkappa(\varkappa-1)V_{\alpha_1}+2f_1\sqrt{h}+\varkappa^2(\varkappa+1)\sqrt{h}]}{12\sqrt{h}\varkappa(\varkappa-1)f_1(f_1+\varkappa^2)-(\varkappa-1)\varkappa^3[6\varkappa(\varkappa-1)V_{\alpha_1}+2f_1\sqrt{h}+\varkappa^2(\varkappa+1)\sqrt{h}]}$$

$$\left(\frac{3\varkappa}{\sqrt{h}}\frac{dV_{\alpha_1}}{d\psi}-\frac{1}{\varkappa-1}\frac{df_1}{d\psi}\right)\psi=\frac{3\varkappa}{\sqrt{h}}V_{\alpha_1}+\frac{f_1}{\varkappa-1}+\frac{\varkappa^2 h}{2} \tag{3.9}$$

and the first integral (3.7).

We see that the first Eq. of (3.9) can be studied separately. It has three singular points

(1) $\left(f_1=-\frac{16}{9}\varkappa^2,\quad V_{\alpha_1}=\frac{272}{81}\frac{\varkappa\sqrt{h}}{\varkappa-1}-\frac{\varkappa h^{3/2}}{6}\right)$

(2) $\left(f_1=-\varkappa^2,\quad V_{\alpha_1}=-\frac{\varkappa\sqrt{h}}{6}\right)$

(3) $\left(f_1=0,\quad V_{\alpha_1}=-\frac{\varkappa h^{3/2}}{6}\right)$

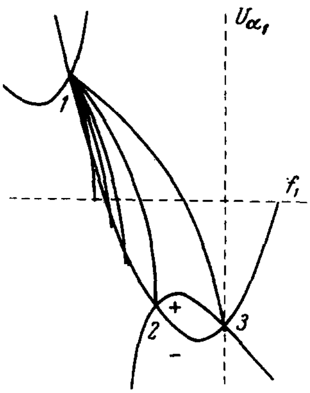


Fig. 3

First of these points correspond to initial data, i.e. to the values of the functions on the characteristic, and it is a node. Let us select, out of all integral curves passing through 1, the one which passes through the second singular point (Fig. 3). At this point the functions have values corresponding to the second term of the asymptotic to the Prandtl-Meyer solution. Along this curve, value of $\psi$ decreases monotonously from $\psi_0$ to 0. Indeed, it is contained within a region bounded by the segments of the isocline $\infty$ connecting points 1 and 2, isocline 0 connecting 2 and 3 and an integral curve given by

$$V_{\alpha_1}=\frac{h^{3/2}f_1^2}{2\varkappa^3(\varkappa+1)}-\sqrt{h}\frac{f}{\varkappa(\varkappa-1)}-\frac{\varkappa h^{3/2}}{6}$$

connecting points 1 and 3. We can see in Fig. 3, that it cannot intersect any of these curves. (Incidentally, we observe that the integral curve connecting points 1 and 3 corresponds to a spherically symmetric solution).

Point 2 is a saddle point. The slope of the required integral curve is given, at this point, by

$$\frac{dV_{\alpha_1}}{df_1}=\frac{-13\sqrt{h}}{3\varkappa(\varkappa-1)}<0$$

and is therefore negative along the whole segment of the curve. Since the integral curve does not intersect the infinity isocline outside the points 1 and 2 , we have

$$(\varkappa-1)\varkappa^2\left(\frac{3\varkappa}{\sqrt{h}}V_{\alpha_1}+\frac{f_1}{\varkappa-1}+\frac{\varkappa^2 h}{2}\right)>6f_1(f_1+\varkappa^2)>0$$

Therefore, on the whole segment of the curve $1\approx 2$,

$$\frac{d\psi}{df_1}=\left(\frac{3\varkappa}{\sqrt{h}}\frac{dV_{\alpha_1}}{df_1}-\frac{1}{x-1}\right)\psi\left(\frac{3\varkappa}{\sqrt{h}}V_{\alpha_1}+\frac{f_1}{\varkappa-1}+\frac{\varkappa^2 h}{2}\right)^{-1}<0$$

Point 2 has the corresponding value $\psi=0$, and in the vicinity of 2 the following asymptotic is valid

$$\begin{aligned} v_\alpha &= \varkappa\sqrt{h}\,\delta+\left(-\frac{1}{6}\varkappa\sqrt{h}+C\psi^{6/7}-\frac{\psi}{\varkappa+1}\right)\delta^3+\ldots \\ v_\delta &= \varkappa+\left(-\frac{\varkappa}{2}+\frac{3}{13\sqrt{h}}C\psi^{6/7}\right)\delta^2+\ldots \\ f &= \varkappa^2-\left(\varkappa^2+\frac{3\varkappa(\varkappa-1)}{13\sqrt{h}}C\psi^{6/7}\right)\delta^2+\ldots \end{aligned} \tag{3.10}$$

where $C$ is a constant of integration. Returning to the point 1 corresponding to the direction of the characteristic $\psi=\psi_0$, we see that the integral curve (1, 3) emerges from this point along a separate branch. Therefore the integral curve in question belongs to a bunch of curves with a common tangent, whose slope is

$$\frac{dV_{\alpha_1}}{df_1}=-\frac{7}{3}\frac{\sqrt{h}}{\varkappa(\varkappa-1)}$$

and, when $\psi$ is almost equal to $\psi_0$, the asymptotic

$$\begin{aligned} v_\alpha &= \varkappa\sqrt{h}\,\delta+\left[\left(\frac{208}{81(\varkappa+1)}-\frac{1}{6}\right)\varkappa h^{3/2}+\frac{41}{8(\varkappa+1)}(\psi-\psi_0)\right]\delta^3+\ldots \\ v_\delta &= \varkappa+\left[\varkappa h\left(\frac{16}{9(\varkappa+1)}-\frac{1}{2}\right)+\frac{21}{8(\varkappa+1)\sqrt{h}}(\psi-\psi_0)\right]\delta^2 \\ f &= \varkappa^2-\left[\frac{16}{9}\varkappa^2+\frac{21}{8}\frac{\varkappa}{h^{3/2}}(\psi-\psi_0)\right]\delta^2 \end{aligned} \tag{3.11}$$

is valid.

**4.** The characteristic $AE$ and the values of gas-dynamic functions on it, were found in Section 2 with help of a spherically symmetric solution. We shall now have to confirm that the segment belonging to $AE$ is a boundary of the region of perturbation. A necessary condition for this is, that gas-dynamic functions must not experience a discontinuity on this segment. Let $\alpha=\varphi(\delta)$ be the equation of $AE$. We shall seek the asymptotic of the solution in the perturbed region near the characteristic

$$\Phi=\Phi_0(\delta)+\Phi_1(\delta)(g-1)+\Phi_2(\delta)(g-1)^2 \quad \text{when} \quad 0\leqslant\delta\leqslant\delta_3,\ g=\alpha/\varphi(\delta) \tag{4.1}$$

where the equation $g=1$ defines the characteristic and $\Phi=(v_\alpha,\ v_\delta,\ f)$ and $\Phi_0(\delta)$ denotes the values which functions assume on the characteristic.

To obtain equations defining the functions $v_{\alpha_1}(\delta)$, $v_{\delta_1}(\delta)$ and $f_1(\delta)$, we insert the expansion (4.1) into the gas-dynamic system (1.6), expand the left-hand sides of these equations in the powers of $(g-1)$ and equate to zero the coefficients of the zero and first power of $(g-1)$. Zero power terms yield only two linearly independent equations. Coefficients of $(g-1)$ when only the terms containing the functions with a subscript 2 are retained in the left-hand side, yield two linear equations for $v_{\alpha_2}$, $v_{\delta_2}$ and $f_2$ and the determinant of this system is equal to zero. To make this system complete, we must equate to zero a linear combination of the right-hand sides dependent only on the functions with the subscript 1 and the relevant coefficients, thus obtaining the required third equation. As a result we obtain the following set of equations defining $v_{\alpha_1}$, $v_{\delta_1}$ and $f_1$:

$$\frac{df_1}{d\delta} = \frac{\zeta(u-\zeta)\sqrt{h}}{2a(cy-ax)}\Big\{-\frac{h(u-\zeta)\zeta}{c(cy-ax)}f_1^2 + \Big[\frac{u\varphi}{\zeta}\Big[\frac{(\varkappa+2)(ax+2cy)}{cy-ax} +$$
$$+\frac{c(a^2+4c^2)(ay+cx)}{a(u-\zeta)^2(cy-ax)} - (\varkappa-1)\frac{(u-\zeta)^2}{a^2} - \frac{c^2}{a^2}\frac{(2c^2-a^2)}{(u-\zeta)^2} - (\varkappa-1)\Big(2+\frac{x(a^2-2c^2)}{a(cy-ax)}\Big)\Big] +$$
$$+\varphi\Big[2+\frac{c}{\zeta(u-\zeta)}\Big(\frac{(\varkappa+1)\cos\gamma - \varphi\sin(\gamma+\sqrt{h}\,\delta)}{(\varkappa+1)\sin\gamma+\varphi\cos(\gamma+\sqrt{h}\,\delta)}a + c\Big)\Big]\Big]f_1 -$$
$$-(\varkappa-1)\Big[\frac{ack\zeta}{u-\zeta}\Big(\frac{a\zeta^2k}{cy-ax}+\varphi\Big) + \frac{c^2(a^2+4c^2)u\varphi ky}{a(u-\zeta)(cy-ax)} + ck\Big[\frac{2(a^2-c^2)}{a}\varphi + ay + cx +$$
$$+\frac{2c\varphi y(\zeta-u)}{a^3(ax+2cy)}(a^2\zeta - 2uc^2 - (\varkappa-1)u(a^2+c^2)) +$$
$$+\frac{ac\zeta^2 - x[(\varphi+y)(\zeta-u)^2 + c(cy-ax) - c(ay+cx)\varphi]}{ax+2cy}\Big] +$$
$$+c^2\varphi\frac{[(\varkappa+1)\cos\gamma-\varphi\sin(\gamma+\sqrt{h}\,\delta)]k + (\varkappa+1)u\zeta^{-1}\sin\gamma}{(\varkappa+1)\sin\gamma+\varphi\cos(\gamma+\sqrt{h}\,\delta)}\Big]\Big\} \qquad (4.2)$$

$$v_{\alpha_1} = \frac{ax-cy}{(\varkappa-1)c\zeta(\zeta-u)}f_1 + kx, \quad v_{\delta_1} = -\frac{ay+cx}{(\varkappa-1)c\zeta(\zeta-u)}f_1 + ky$$
$$k = \frac{u\varphi}{a\zeta^3}(ax+2cy)$$

Notation used here is that adopted in Section 2. Functions appearing in the right-hand sides are obtained by integration of the system (2.1) and (2.2). From the asymptotic (3.11) it follows that when $f_1(0) = 0$ and $\delta$ are small

$$f_1 \approx -28/27\,\varkappa^2\delta^2$$

First Eq. of (4.2) is a Riccati equation; a partial integral of this equation is well known, and it corresponds to the spherically symmetric solution

$$F_1 = -\frac{2(\varkappa-1)uc^2y\varphi(\zeta-u)}{a^2\zeta^2} \qquad (4.3)$$

which can, therefore, be reduced to a linear equation

$$\frac{dz}{d\delta} = -\frac{\zeta(u-\zeta)\sqrt{h}}{2a(cy-ax)}\Big\{\Big\{\frac{u\varphi}{\zeta}\Big[\frac{6c^2}{a^2+c^2} - 5\varkappa + 4 - 5(\varkappa+1)\frac{c^2}{a^2} -$$
$$-2x\frac{(\varkappa+1)c^2+(\varkappa-2)a^2}{a(cy-ax)}\Big] + \varphi\Big[2+\frac{c^2}{\zeta(u-\zeta)} + \frac{ac}{\zeta(u-\zeta)}\times$$
$$\times\frac{(\varkappa+1)\cos\gamma - \varphi\sin(\gamma+\sqrt{h}\,\delta)}{(\varkappa+1)\sin\gamma+\varphi\cos(\gamma+\sqrt{h}\,\delta)}\Big]\Big\}z - \frac{h(u-\zeta)\zeta}{c(cy-ax)}\Big\} \qquad (4.4)$$

where

$$z = \Big(f_1 + \frac{2(\varkappa-1)c^2u(\zeta-u)y\varphi}{a^2\zeta^2}\Big)^{-1}$$

When $\gamma < \gamma_0$, then a point $D(\delta = \delta_2)$ exists on $AE$

$$u = 0, \qquad a^2 = (\zeta-u)^2 - c^2 = 0$$

and in its vicinity we have

$$a^2 \sim u\ln u, \qquad a \approx \sqrt{h}\varphi\,(\zeta^2/cy)(\delta-\delta_2)$$

Consequently, retaining the principal terms in (4.4), we obtain

$$\frac{dz}{d\delta} + \frac{1}{2}\frac{z}{\delta-\delta_2} + \frac{h}{2y\varphi(\delta-\delta_2)} = 0 \qquad (4.5)$$

General integral of this equation is

$$z = -h / y\varphi + c(\delta - \delta_2)^{1/2}$$

hence

$$z = -\frac{h}{y\varphi}, \qquad f_1 = -\frac{y\varphi}{h} \quad \text{when} \quad \delta = \delta_2$$

Thus, any integral curve of the first Eq. of (4.2) which can be taken up to the value of $\delta = \delta_2$ except the one representing the partial integral (4.3) assumes, at $\delta = \delta_2$, a value $f_1 = -y\varphi/h < 0$. Taking into account that near the point $\delta = 0$

$$f_1 < F_1 \qquad (f_1 \approx -{}^{28}/_{27}\varkappa^2\delta^2, \ F_1 \approx -{}^{8}/_{9}\varkappa^2\delta^2)$$

we conclude that a value $\delta_3$ exists on the interval $0 < \delta < \delta_2$, at which $f_1$ becomes infinite.

From the system (4.2) it follows that, when $\delta \to \delta_3$, then the functions with the subscript 1 appearing in (4.1) increase like $(\delta - \delta_3)^{-1}$ or, more accurately,

$$\Phi_1 \approx \frac{\Phi_{10}}{\delta - \delta_3} + \Phi_{11}$$

and

$$f_{10} = \left[\frac{2ac}{h^{3/2}}\left(\frac{cy - ax}{\zeta(\zeta - u)}\right)^2\right]_{\delta=\delta_3}, \qquad v_{\alpha_{10}} = \left[\frac{2a}{(\varkappa + 1)\sqrt{h}}\left(\frac{ax - cy}{\zeta(\zeta - u)}\right)^3\right]_{\delta=\delta_3} \tag{4.6}$$

$$v_{\delta_{10}} = \left[\frac{2a(ax - cy)^2(ay + cx)}{(\varkappa + 1)\sqrt{h}\,\zeta^3(u - \zeta)^3}\right]_{\delta=\delta_3}$$

(Formulas for $\Phi_{11}$ are too unwieldy and shall not be given here). Similarly we can obtain equations for the functions with the subscript 2. Their asymptotics, when $\delta \to \delta_3$, will have the form

$$\Phi_2 \approx \frac{\Phi_{20}}{(\delta - \delta_3)^3} + \frac{\Phi_{21}}{(\delta - \delta_3)^2} \tag{4.7}$$

We note that the values of $\Phi_{20}$ can only be obtained by integration. Thus, when $\delta < \delta_3$ and $(\alpha - \varphi)/(\delta - \delta_3)^2$ is sufficiently small, then the following asymptotic is valid:

$$\Phi \approx \Phi_0(\delta) + \left[\frac{\Phi_{10}}{\delta - \delta_3} + \Phi_{11}\right]\frac{\alpha - \varphi}{\varphi} + \left[\frac{\Phi_{20}}{(\delta - \delta_3)^3} + \frac{\Phi_{21}}{(\delta - \delta_3)^2}\right]\left(\frac{\alpha - \varphi}{\varphi}\right)^2 + \ldots \tag{4.8}$$

The above analysis is valid when the point $[\delta_3, \varphi(\delta_3)]$ does not lie on the axis of symmetry. We find it can reach the axis of symmetry only when $\gamma > \gamma_0$. More accurately, for each $\varkappa$ there exists a value $\gamma_\varkappa > \gamma_0$, such that when $\gamma > \gamma_\varkappa$ then the point $[\delta_3, \varphi(\delta_3)]$ falls on the axis of symmetry and the asymptotic will, at this point, be given by

$$\Phi_1 \approx \frac{\psi_{10}}{\sqrt{\delta - \delta_3}} \tag{4.9}$$

**5.** Since the point $[\delta_3, \varphi(\delta_3)]$ is a singular point of our solution, we ougth to study the behavior of the gas-dynamic functions in the whole neighborhood of this point. Before all, we shall note that in the region where the asymptotic (4.8) is valid, the characteristics belonging to the same family as $\alpha = \varphi(\delta)$ all converge at the point $[\delta_3, \varphi(\delta_3)]$. Equation of this family of characteristics in terms of the variables $\alpha$ and $\delta$, is

$$\frac{d\alpha}{d\delta} = -\sqrt{h}\,\alpha\,\frac{(v_\alpha - \alpha)^2 - f}{f\,[v_\delta^2 + (v_\alpha - \alpha)^2 - f]^{1/2} - v_\delta(v_\alpha - \alpha)} \tag{5.1}$$

We can, within the indicated region, replace the functions entering this equation with their asymptotic representations given by (4.8). As a result we obtain

$$\frac{d(\alpha - \varphi)}{d\delta} \approx \frac{\alpha - \varphi}{\delta - \delta_3} \tag{5.2}$$

i.e., $\alpha - \varphi \approx C(\delta - \delta_3)$, or in other words the characteristics pass through the point

$(\delta_3, \varphi(\delta_3))$. Consequently, a shock wave is formed at this point and we cannot describe it without investigating the whole neighborhood of this point in more detail.

Formulas (4.8) can be rewritten thus

$$\Phi = \Phi_0(\delta_3) + \left[\Phi'_0(\delta_3) + \frac{\Phi_{10}}{\varphi(\delta_3)}\frac{\alpha - \varphi(\delta)}{(\delta - \delta_3)^2} + \frac{\Phi_{20}}{\varphi^2(\delta_3)}\left(\frac{\alpha - \varphi(\delta)}{(\delta - \delta_3)^2}\right)^2\right](\delta - \delta_3) +$$
$$+ \left[\Phi_0''(\delta_3) + \left(\frac{\Phi_{11}}{\varphi(\delta_3)} - \frac{\Phi_{10}\varphi'(\delta_3)}{\varphi^2(\delta_3)}\right)\frac{\alpha - \varphi(\delta)}{(\delta - \delta_3)^2} + \left(\frac{\Phi_{21}}{\varphi^2(\delta_3)} - \frac{2\Phi_{20}\varphi'(\delta_3)}{\varphi^3(\delta_3)}\right) \times\right.$$
$$\left. \times \left(\frac{\alpha - \varphi(\delta)}{(\delta - \delta_3)^2}\right)^2\right](\delta - \delta_3)^2 + \ldots \tag{5.3}$$

therefore it seems feasible to seek the solutions near this point in the form (5.4)

$$F = F_0(\chi) + (\delta - \delta_3) F_1(\chi) + (\delta - \delta_3)^2 F_2(\chi) + \ldots \quad \left(\chi = \frac{\alpha - \varphi}{(\delta - \delta_3)^2}, \quad F = (v_\alpha, v_\delta, f)\right)$$

The following three homogeneous differential equations define the functions bearing the subscript 0

$$\left(v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right) v_{\alpha_0}' + \frac{1}{\varkappa - 1} f_0' = 0$$
$$\left(v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right) v_{\delta_0}' - \frac{1}{\varkappa - 1} \frac{\varphi'}{\sqrt{h}\varphi} f_0' = 0 \tag{5.5}$$
$$\left(v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right) f_0' + (\varkappa - 1) f_0 \left(v_{\alpha_0}' - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}'\right) = 0$$

Relevant initial data are

$$v_{\alpha_0} = \varphi + \left(1 - \frac{u}{\zeta}\right) y, \qquad v_{\delta_0} = \left(1 - \frac{u}{\zeta}\right) x, \qquad f_0 = c^2 \quad \text{when} \quad \chi = 0$$

and all magnitudes appearing here without a subscript denote the values of corresponding functions of a spherical self-similar solution at $\delta = \delta_3$ and $\alpha = \varphi(\delta_3)$. Constants corresponding to initial data will constitute a solution of (5.5) and its determinant will then become zero. Functions with the subscript 1 will have a homogeneous system whose determinant will be identical with the previous one and again equal to zero. Initial data for this system are defined by the asymptotic (5.3). Obviously, the constants with initial data will satisfy this system, which consists of only two linearly independent equations. The remaining equation can be obtained by making use of the fact that when the subscript is equal to $i + 1$ ($i \geq 1$) then nonhomogeneous equations are obtained and their determinant is equal to zero. Consequently there exists a linear relationship between the right-hand sides of these equations and a system defining the functions with the subscript $i$ supplemented by this linear relation, has a solution which coincides with (5.3). Thus, for functions with the subscript 1 we have

$$\left(v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right) v_{\alpha_1}' + \frac{1}{\varkappa - 1} f_1' = 0, \left(v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right) v_{\delta_1}' -$$
$$- \frac{\varphi'}{\sqrt{h}\varphi(\varkappa - 1)} f_1' = 0, \quad f_1' = \frac{1}{2}\left(f_1 - \frac{df}{d\delta}\right)\left[\chi - \frac{\varphi}{2f_{10}}\left(f_1 - \frac{df}{d\delta}\right)\right]^{-1} \tag{5.6}$$

and a solution which approaches (5.3) as $\chi \to 0$, is given by

$$f_1 = \frac{f_{10}^2}{2f_{20}}\left[1 - \left(1 - \frac{4f_{20}}{f_{10}\varphi}\chi\right)^{1/2}\right] + \frac{df}{d\delta} \tag{5.7}$$

$$\left[v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right] v_{\alpha_1} = -\frac{1}{\varkappa - 1} f_1 + \left(v_{\alpha_0} - \varphi - \frac{\varphi'}{\sqrt{h}\varphi} v_{\delta_0}\right)\frac{dv_\alpha}{d\delta} +$$

$$+\frac{1}{\varkappa-1}\frac{df}{d\delta},\left[v_{\alpha_0}-\varphi-\frac{\varphi'}{\sqrt{h}\,\varphi}v_{\delta_0}\right]v_{\delta_1}=\frac{\varphi'}{(\varkappa-1)\sqrt{h}\,\varphi}f_1+$$
$$+\left(v_{\alpha_0}-\varphi-\frac{\varphi'}{\sqrt{h}\,\varphi}v_{\delta_0}\right)\frac{dv_\delta}{d\delta}-\frac{\varphi'}{(\varkappa-1)\sqrt{h}\,\varphi}\frac{df}{d\delta}$$

Functions with a subscript 2 can be obtained in a similar manner. Having found $F_0(\chi)$, $F_1(\chi)$ and $F_2(\chi)$ we can obtain the asymptotic of the wave equation as well as those of the values of functions on this wave near the point ($\delta_3$, $\varphi(\delta_3)$). Let $F^+(\chi,\delta)$ denote the values of functions on the wave, and $F^-(\chi, \delta)$ before the wave and let the latter correspond to a spherically symmetric solution which can be written near the point $\delta_3$, $\varphi(\delta_3)$ in the form (5.4). For example,

$$f^-=f_0^-+\frac{df_0^-}{d\delta}(\delta-\delta_3)+\left[\frac{1}{2}\frac{d^2f_0^-}{d\delta^2}+\frac{F_1(\delta_3)}{\varphi}\chi\right](\delta-\delta_3)^2+\ldots \tag{5.8}$$

Here the functions and their derivatives are taken on the characteristic $\alpha=\varphi(\delta)$ at the point $\delta=\delta_3$ and $F_1(\delta_3)$ is given by (4.3). We can write formulas for $v_\alpha^-$ and $v_\delta^-$ in the same manner. Wave equations shall be sought in the form

$$\chi=\chi_b+c_b\ (\delta-\delta_3) \tag{5.9}$$

and conditions on the shock wave will be

$$\rho^+(D-u^+)=\rho^-(D-u^-),\ \ p^++\rho^+(D-u^+)^2=p^-+\rho^-(D-u^-)^2,\ \ w^+=w^- \tag{5.10}$$

Denoting the angle between the tangent to the wave and the horizontal by $\beta$, we obtain the normal $u$ and tangential $w$ velocities as follows:

$$u=-v_\alpha\sin[(\gamma+\sqrt{h}\,\delta)-\beta]-v_\delta\cos[(\gamma+\sqrt{h}\,\delta)-\beta]+(\varkappa+1)\cos(\gamma-\beta)$$
$$w=v_\alpha\cos[(\gamma+\sqrt{h}\,\delta)-\beta]-v_\delta\sin[(\gamma+\sqrt{h}\,\delta)-\beta]-(\varkappa+1)\sin(\gamma-\beta) \tag{5.11}$$

Let $\alpha=\alpha(\delta)$ be the wave equation, then

$$\operatorname{tg}\beta=\frac{\sin(\gamma+\sqrt{h}\,\delta)\,d\alpha/d\delta+\cos(\gamma+\sqrt{h}\,\delta)\,\alpha\sqrt{h}}{\cos(\gamma+\sqrt{h}\,\delta)\,d\alpha/d\delta-\sin(\gamma+\sqrt{h}\,\delta)\,\alpha\sqrt{h}} \tag{5.12}$$

and consequently

$$\alpha\sqrt{h}\operatorname{ctg}(\gamma+\sqrt{h}\,\delta-\beta)=-d\alpha/d\delta \tag{5.13}$$

Using relations obtained from the equation of state $p=\varkappa\rho^\varkappa$

$$\rho=\left(\frac{f}{\varkappa^2}\right)^{1/\varkappa-1}\qquad p=\varkappa\left(\frac{f}{\varkappa^2}\right)^{\varkappa/\varkappa-1}$$

we can write the second and third condition on thr wave as

$$\frac{d\alpha}{d\delta}=\frac{v_\delta^+-v_\delta^-}{v_\alpha^+-v_\alpha^-}\alpha\sqrt{h},\quad [v_\delta^+-v_\delta^-]^2+[v_\alpha^+-v_\alpha^-]^2=\frac{f^+}{\varkappa}\left[\left(\frac{f^+}{f^-}\right)^{\varkappa/(\varkappa-1)}-1\right]\times$$
$$\times\left[1-\left(\frac{f^+}{f^-}\right)^{-1/(\varkappa-1)}\right] \tag{5.14}$$

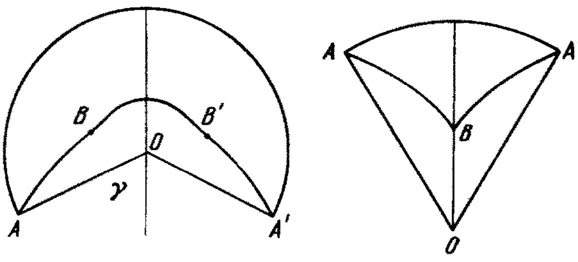
Obviously, $\delta=\delta_3+(\delta-\delta_3)$, therefore near ($\delta_3$, $\varphi(\delta_3)$) the wave equation has, by (5.9), the following asymptotic

$$\alpha=\varphi(\delta_3)+\varphi'(\delta_3)(\delta-\delta_3)+\left[\chi_b+\frac{1}{2}\varphi''(\delta_3)\right](\delta-\delta_3)^2+\ldots$$
$$\frac{d\alpha}{d\delta}=\varphi'(\delta_3)+[2\chi_b+\varphi''(\delta_3)](\delta-\delta_3) \tag{5.15}$$

Formulas (5.4) along the wave are written as

$$F=F_0(\chi_b)+(\delta-\delta_3)F_1(\chi_b)+(\delta-\delta_3)^2[c_bF_1'(\chi_b)+F_2(\chi_b)]+\ldots \tag{5.16}$$

Inserting into (5.14) the asymptotics (5.8), (5.15) and (5.16) and equating the coefficients of like powers of $(\delta - \delta_3)$, we obtain two equations for $\chi_b$ and $\sigma_b$. We note that Formulas (5.14) cease to be identities only after the substitution of three terms of (5.8) and (5.16) and of two terms of (5.15), respectively. Thus we obtain a wave of an infinitesimal amplitude near $(\delta_3, \varphi(\delta_3))$. We cannot, for obvious reasons, define the wave everywhere between the point $\delta_3, \varphi(\delta_3)$ and the axis of symmetry, using the above asymptotic method. To do this, we would have to integrate partial differential equations numerically. We can expect, of course, that the wave front will never depart very far from the characteristic $\alpha = \varphi(\delta)$. On the symmetry axis, the tangent to the shock wave front is horizontal. Indeed, the tangential velocity $w$ is given in terms of $v_r$ and $v_z$ (see Section 1) as follows:




a Fig. 4 b

$$w = v_r \cos\beta + v_z \sin\beta, \qquad u = v_r \sin\beta - v_z \cos\beta \tag{5.17}$$

Since on the symmetry axis we have $v_r = 0$, conditions at the wave front at this point yield

$$(v_z^+ - v_z^-)\sin\beta = 0, \qquad (p^+ - p^-)\left(\frac{1}{\rho^-} - \frac{1}{\rho^+}\right) = (v_z^+ - v_z^-)^2 \cos^2\beta \tag{5.18}$$

When the wave has a nonzero amplitude, then these conditions hold only, when $\sin\beta = 0$.

Thus, the region of coincidence of our solution with the spherically symmetric solution is bounded (when $\gamma < \gamma_\varkappa$) by the segments of characteristics $\alpha = \varphi(\delta)$ joining the points $A$ and $A'$ situated symmetrically with respect to the $\eta$-axis, with another symmetric pair of points $B$ and $B'(\delta_3, \varphi(\delta_3))$, by the shock wave front $BB'$ and by the detonation wave front $AA'$ (Fig. 4a).

N o t e. In the approximation obtained, the line $\chi = f_{10}\varphi / 4f_{20}$ is a characteristic belonging to the same family as $\alpha = \varphi(\delta)$. Perturbed area in the vicinity of the point $[\delta_3, \varphi(\delta_3)]$ is described in terms of the values of $\chi$ varying in the direction $0 \to -\infty \to \chi_b$. If the characteristic $\chi = f_{10}\varphi / 4f_{20}$ belongs to the perturbed area, then, in the area situated behind this characteristic the right-hand sides of the asymptotic Formulas (5.7) become complex when $(4f_{20} / f_{10}\varphi)\chi > 1$. This is apparently connected with the existence of a region of subsonic flow behind the wave and in this case solution can be obtained only by performing the integration of partial differential equations.

When $\gamma > \gamma_\varkappa$, the characteristics $\alpha = \varphi(\delta)$ emerging from $A$ and $A'$ meet on the $\eta$-axis at the point $B$ at an angle $> \pi$ (Fig. 4b). From the asymptotic equation (4.9) it follows that the characteristics belonging to the same family do not intersect, therefore no shock wave is formed at $B$. Using the same asymptotic (4.9) we shall seek a solution near the point $B$ in the form

$$F = F_0 + F_1(l) \mid \delta - \delta_3 \mid^{1/2}, \quad l = \frac{\alpha - \varphi}{\delta - \delta_3} \tag{5.19}$$

where $\delta_3$ is the value of $\delta$ corresponding to the point $B$. The following value corresponds to the $\eta$-axis

$$l = l_0 = \frac{-\sqrt{h}\,\varphi(\delta_3)\, c\,(x^2 + y^2)}{x\,(ax - cy)} \tag{5.20}$$

Radial velocity $v_r$ is equal to zero on the $\eta$-axis, i.e.

$$v_\alpha \cos(\gamma + \sqrt{h}\,\delta) - v_\delta \sin(\gamma + \sqrt{h}\,\delta) + (\varkappa + 1)\sin\gamma = 0 \tag{5.21}$$

and this is automatically fulfilled for the values of functions on the characteristic $\alpha = \varphi(\delta)$ at the point $\delta = \delta_3$ (functions with the subscript 0). For functions with the subscript 1, we require that

$$xv_{\alpha_1}(l_0) - yv_{\delta_1}(l_0) = 0 \tag{5.22}$$

and we have for these functions the following system:

$$\left(1 - \frac{u}{\zeta}\right)\left[y - x\frac{\varphi' + l}{\sqrt{h}\,\varphi}\right]v_{\alpha_1}' + \frac{1}{\varkappa - 1}f_1' = -\frac{(1 - u/\zeta)\,x}{2\sqrt{h}\,\varphi}v_{\alpha_1}$$

$$\left(1 - \frac{u}{\zeta}\right)\left[y - x\frac{\varphi' + l}{\sqrt{h}\,\varphi}\right]v_{\delta_1}' - \frac{1}{\varkappa - 1}\frac{\varphi' + l}{\sqrt{h}\,\varphi}f_1' =$$

$$= -\frac{1}{2\sqrt{h}\,\varphi}\left[\left(1 - \frac{u}{\zeta}\right)xv_{\delta_1} + \frac{f_1}{\varkappa - 1}\right]$$

$$\left(1 - \frac{u}{\zeta}\right)\left[y - x\frac{\varphi' + l}{\sqrt{h}\,\varphi}\right]f_1' - (\varkappa - 1)c^2\left[\frac{\varphi' + l}{\sqrt{h}\,\varphi}v_{\delta_1}' - v_{\alpha_1}'\right] =$$

$$= -\frac{1}{2\sqrt{h}\,\varphi}\left[\left(1 - \frac{u}{\zeta}\right)xf_1 + (\varkappa - 1)c^2 v_{\delta_1}\right] - (\varkappa - 1)c^2\frac{v_{\alpha_1} - (y/x)\,v_{\delta_1}}{\zeta + \varphi' - \sqrt{h}\,\varphi y/x} \tag{5.23}$$

The characteristic $\alpha = \varphi(\delta)$ has a corresponding line $l = 0$. Point ($l = 0$, $F = F_0$) is a singular point of the system (5.23) and one can emerge from it along the asymptotic (4.9) which can be rewritten as follows:

$$l \to 0, \quad F \approx F_0 + (\psi_{10}/\varphi(\delta_3))\, l\, (\delta - \delta_3)^{1/2} \tag{5.24}$$

System (5.23) has a first integral which, with the asymptotic (5.24) taken into account, can be written as

$$(1 - u/\zeta)\,[v_{\alpha_1}y + v_{\delta_1}x] + f_1/(\varkappa - 1) = 0 \tag{5.25}$$

Asymptotic of the solution as $l \to l_0$, is

$$v_{\alpha_1}y + v_{\delta_1}x \approx A, \quad v_{\alpha_1}x - v_{\delta_1}y \approx B\,|l - l_0|^{-1} + \frac{a^2x^2A}{4\pi\sqrt{h}\,\varphi c^2(x^2 + y^2)}(l - l_0)$$

$$(A, B = \text{const}) \tag{5.26}$$

Point $l = l_1$ given by

$$l_1 = -\frac{2\sqrt{h}\,ac\varphi\,(x^2 + y^2)}{(ax - cy)(ax + cy)}$$

is also a singular point of (5.23) and corresponds to a characteristic which belongs to a family of which $\alpha = \varphi(\delta)$ is not a member. At this point, the functions are connected by the following relations:

$$v_{\alpha_{10}}(ay + 3cx) = v_{\delta_{10}}(3cy - ax) \tag{5.27}$$

As $l \to l_1$, we have the following asymptotic

$$v_{\alpha_1} \approx v_{\alpha_{10}} + A_1(l - l_1), \qquad v_{\delta_1} \approx v_{\delta_{10}} + B_1(l - l_1) \tag{5.28}$$

where $A_1$ and $B_1$ are connected by a single linear relation

$$\frac{ay - cx}{ax + cy}A_1 + B_1 = \frac{v_{\alpha_{10}}}{2\sqrt{h}\,\varphi} \tag{5.29}$$

From (5.22) it follows that $B$ in (5.26) must be equal to zero. Taking into account that $l$ varies in the direction $0 \to \infty$, $-\infty, \to l_1 \to l_0$ we can show this by varying $A_1$ and $B_1$ within the limits imposed by (5.29), i.e. allowing a weak discontinuity on passing along the characteristic $l = l_1$.

Problems touched upon in the present paper were discussed with S. K. Godunov. Many of the formulas quoted were checked by I. L. Kireeva who also, together with N. I. Kurancheva, performed numerical integration of equations on a computer. The author wishes to take this opportunity to express his gratitude to all persons mentioned above.

BIBLIOGRAPHY

1. Zel'dovich, Ia. B., On pressure and velocity distribution in explosion products when the detonation wave expands with a spherical symmetry. ZhEFT, Vol. 12, No. 9, 1942.

Translated by L. K.

# AN ANALYTIC SOLUTION OF A SYSTEM OF EQUATIONS FOR THE AXISYMMETRICAL FLOW OF AN INCOMPRESSIBLE FLUID

PMM Vol. 1, No. 5, 1967, pp. 928-931

G. I. NAZAROV and A. A. PUCHKOV
(Tomsk)

(Received April 4, 1967)

In contrast to the solution of Garabedian [1] and to the solution constructed by the Bergman method [2], we derive an exact general solution for the pair of functions $\varphi$ and $\psi$ ($\varphi$ is the velocity potential, $\psi$ is the stream function) of a system of partial differential equations describing the axisymmetrical flow of an incompressible ideal fluid. Our solution depends on an arbitrary analytic function of a complex variable and is bounded on the axis of symmetry.

The solutions constructed in [1] and [2] increase without limit as the axis of symmetry is approached.

Three-dimensional steady-state axisymmetrical flows of an incompressible fluid are described by the system of Eqs. [3]

$$\frac{\partial\varphi}{\partial x} = -\frac{1}{y}\frac{\partial\psi}{\partial y}, \qquad \frac{\partial\varphi}{\partial y} = \frac{1}{y}\frac{\partial\psi}{\partial x} \tag{1}$$

Here the velocity potential $\varphi$ and the stream function $\psi$ depend on only the two variables $x$, $y$ of the cylindrical coordinate system ($y > 0$ and $x$ is parallel to the axis of symmetry).

The integrals of system (1) will be sought in series form

$$\varphi = \Omega(y) + \sum_{k=0}^{\infty} \alpha_k(y)\frac{\partial^k\Phi}{\partial y^k}, \qquad \psi = A + Bx + \sum_{k=0}^{\infty} \beta_k(y)\frac{\partial^k\Psi}{\partial y^k} \tag{2}$$

Here $\Phi$, $\Psi$ are arbitrary harmonic functions which satisfy the Cauchy-Riemann conditions

$$\frac{\partial\Phi}{\partial x} = \frac{\partial\Psi}{\partial y}, \qquad \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x} \tag{3}$$

where $\Omega, \alpha_k, \beta_k$ $(k = 0, 1, 2, \ldots)$ are the required functions of the single argument $y$.

Let us construct the corresponding derivatives of (2), substitute them into (1), and recall Eqs. (3) and relations of the form

$$\frac{\partial^{k+1}\Phi}{\partial x\partial y^k} = \frac{\partial^{k+1}\Psi}{\partial y^{k+1}}, \qquad \frac{\partial^{k+1}\Psi}{\partial x\,\partial y^k} = -\frac{\partial^{k+1}\Phi}{\partial y^{k+1}} \tag{4}$$